Coupled Harmonic Oscillators and Feynman's Rest of the Universe

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Abstract

According to Feynman, the universe consists of two parts - the system in which we are interested and the rest of the universe which our measurement process does not reach. Feynman then formulates the density matrix in terms of the observable world and the rest of the universe. It is shown that coupled harmonic oscillators can serve as an illustrative example for Feynman's "rest of the universe." It is pointed out that this simple example has far-reaching consequences in many branches of physics, including statistical mechanics, measurement theory, information theory, thermo-field dynamics,

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quantum optics, and relativistic quantum mechanics. It is shown that our ignorance of the rest of the universe increases the uncertainty and entropy in the system in which we are interested.

I. INTRODUCTION

Because of its mathematical simplicity, the harmonic oscillator provides soluble models in many branches of physics. It often gives a clear illustration of abstract ideas. In his book on statistical mechanics [1], Feynman makes the following statement on the density matrix. *When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system.*

The purpose of this paper is to study Feynman's rest of the universe and related problems of current interest using a pair of coupled oscillators. Starting from the classical mechanics of harmonic oscillators, we formulate the symmetry of the oscillator system in terms of the group Sp(2) which is of current interest and which is locally isomorphic to the (2 + 1)-dimensional Lorentz group. This symmetry is then extended to the quantum mechanics of coupled oscillators. This allows us to study measurable and unmeasurable variables in terms of the two oscillator coordinates. The unmeasurable variable constitutes Feynman's rest of the universe. We shall study the effects of this unmeasurable variable to the measurable variable in terms of the uncertainty relation and entropy.

In Sec. II, we reformulate the classical mechanics of two coupled oscillators in terms of the Sp(2) group. The symmetry operations include rotations and squeezes in the two-dimensional coordinate system of two oscillator coordinates. In Sec. III, this symmetry property is extended to the quantum mechanics of the coupled oscillators. In Sec. IV, we use the Wigner phase-space distribution function to see the effect of the unobservable

variable on the uncertainty relation in the observable world. In Sec. V, we use the density matrix to study the entropy of the system due to our ignorance of the unobservable variable. In Sec. VI, it is shown that the system of two coupled oscillators can serve as an analog computer for many of the physical theories and models of current interest. Section VII contains some concluding remarks.

II. COUPLED OSCILLATORS IN CLASSICAL MECHANICS

Two coupled harmonic oscillators serve many different purposes in physics. It is widely believed that this oscillator problem can be formulated into a problem of quadratic equation of two variables, and the quadratic equation can be separated by a simple rotation. It is true that the problem can be reduced to a quadratic equation, but it is not true that this equation can be solved by one rotation. Indeed, in order to understand fully this simple problem, we have to employ the SL(4,r) group which is isomorphic to the Lorentz group O(3,3) with fifteen parameters [2].

In this paper, we do not need all the symmetries available from the O(3,3) group. We shall use one of the O(2,1)-like subgroups which is by now a standard language in physics in all branches of physics. Let us consider a system of coupled oscillators. The Hamiltonian for this system is

$$H = \frac{1}{2} \left\{ \frac{1}{m_1} p_1^2 + \frac{1}{m_2} p_2^2 + A x_1^2 + B x_2^2 + C x_1 x_2 \right\}. \tag{2.1}$$

By making scale changes of x_1 and x_2 to $(m_2/m_1)^{1/4} x_1$ and $(m_1/m_2)^{1/4} x_2$ respectively, it is possible to write the above Hamiltonian in the form [3,4]

$$H = \frac{1}{2m} \left\{ p_1^2 + p_2^2 \right\} + \frac{1}{2} \left\{ A x_1^2 + B x_2^2 + C x_1 x_2 \right\}, \tag{2.2}$$

with $m = (m_1 m_2)^{1/2}$. We can decouple this Hamiltonian by making the coordinate transformation:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.3}$$

Under this rotation, the kinetic energy portion of the Hamiltonian in Eq.(2.3) remains invariant. Thus we can achieve the decoupling by diagonalizing the potential energy. Indeed, the system becomes diagonal if the angle α becomes

$$\tan \alpha = \frac{C}{B - A}. \tag{2.4}$$

This diagonalization procedure is well known. What is new in this note is to introduce the new parameters K and η defined as

$$K = \sqrt{AB - C^2/4},$$

$$\exp(-\eta) = \frac{A + B + \sqrt{(A - B)^2 + C^2}}{4AB - C^2}. \tag{2.5}$$

In terms of this new set of variables, the Hamiltonian can be written as

$$H = \frac{1}{2m} \left\{ p_1^2 + p_2^2 \right\} + \frac{K}{2} \left\{ e^{2\eta} y_1^2 + e^{-2\eta} y_2^2 \right\}, \tag{2.6}$$

with

$$y_1 = x_1 \cos \frac{\alpha}{2} - x_2 \sin \frac{\alpha}{2},$$

$$y_2 = x_1 \sin \frac{\alpha}{2} + x_2 \cos \frac{\alpha}{2}. \tag{2.7}$$

In this way, we can study the symmetry properties of the coupled oscillators systematically using group theoretical methods. The coordinate rotation of Eq.(2.3) is generated by

$$J_0 = -\frac{i}{2} \left\{ x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right\}. \tag{2.8}$$

The η variable changes the scale of y_1 in one direction while changing that of y_2 in the opposite direction. If y_1 is expanded then y_2 becomes contracted so as to preserve the product $y_1 y_2$. This is called the squeeze transformation. The squeeze operation which changes the scales of y_1 and y_2 is generated by

$$S_y = -\frac{i}{2} \left\{ y_2 \frac{\partial}{\partial y_2} - y_1 \frac{\partial}{\partial y_1} \right\}. \tag{2.9}$$

From the linear transformation of Eq.(2.3), this expression can be written as

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$$S_y = S_1 \cos\alpha - S_2 \sin\alpha, \tag{2.10}$$

with

$$S_1 = -\frac{i}{2}\left\{x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}\right\},$$

$$S_2 = -\frac{i}{2}\left\{x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_1}\right\}. \tag{2.11}$$

Indeed, the generators J_0, S_1 and S_2 satisfy the commutation relations:

$$[J_0, S_1] = iS_2, \quad [J_0, S_2] = -iS_1, \quad [S_1, S_2] = iJ_0, \tag{2.12}$$

This set of commutation relations is identical to that for the group SU(1,1) which is locally isomorphic to the (2 + 1)-dimensional Lorentz group [4]. If the problem is extended to the four-dimensional phase space consisting of the $x_1, x_2, p_1,$ and p_2 variables, the symmetry group is Sp(4) which is locally isomorphic to O(5,2) [4–6]. These groups are known to provide the standard language for the one-mode and two-mode squeezed states respectively [4], in addition to their traditional roles in other branches of physics, including classical mechanics, nuclear, elementary particle and condensed matter physics.

III. QUANTUM MECHANICS OF COUPLED OSCILLATORS

If y_1 and y_2 are measured in units of $(mK)^{1/4}$, the ground-state wave function of this oscillator system is

$$\psi_\eta(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}(e^\eta y_1^2 + e^{-\eta} y_2^2)\right\}. \tag{3.1}$$

The wave function is separable in the y_1 and y_2 variables. However, for the variables x_1 and x_2, the story is quite different.

The key question is how the measurement or non-measurement of one variable affects the world of the other variable. If we are not able to make any measurement in the x_2 space, how does this affect the quantum mechanics in the x_1 space. This effect is not trivial. Indeed, the x_2 space in this case corresponds to Feynman's rest of the universe, if we only

know how to do quantum mechanics in the x_1 space. We shall discuss in this paper how we can carry out a quantitative analysis of Feynman's rest of the universe.

Let us write the wave function of Eq.(3.1) in terms of x_1 and x_2, then

$$\psi_\eta(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}\left[e^\eta(x_1 \cos\frac{\alpha}{2} - x_2 \sin\frac{\alpha}{2})^2\right.\right.$$
$$\left.\left. + e^{-\eta}(x_1 \sin\frac{\alpha}{2} + x_2 \cos\frac{\alpha}{2})^2\right]\right\}. \tag{3.2}$$

If $\eta = 0$, this wave function becomes

$$\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}(x_1^2 + x_2^2)\right\}. \tag{3.3}$$

For other values of η, the wave function of Eq.(3.2) can be obtained from the above expression by a unitary transformation generated by the operators given in Eq.(2.8) and Eq.(2.11). We should then be able to write Eq.(3.2) as

$$\sum_{m_1 m_2} A_{m_1 m_2}(\alpha, \eta)\phi_{m_1}(x_1)\phi_{m_2}(x_2), \tag{3.4}$$

where $\phi_m(x)$ is the mth excited-state oscillator wave function. The coefficients $A_{m_1 m_2}(\eta)$ satisfy the unitarity condition

$$\sum_{m_1 m_2} |A_{m_1 m_2}(\alpha, \eta)|^2 = 1. \tag{3.5}$$

It is possible to carry out a similar expansion in the case of excited states.

The question then is what lessons we can learn from the situation in which we are not able to make measurements on the x_2 variable. In order to study this problem, we use the density matrix and the Wigner phase-space distribution function.

IV. WIGNER FUNCTION AND UNCERTAINTY RELATION

In his book, Feynman raises the issue of the rest of the universe in connection with the density matrix. Indeed, the density matrix plays the essential role when we are not able to measure all the variables in quantum mechanics [7,8]. In the present case, we assume that

we are not able to measure the x_2 coordinate. It is often more convenient to use the Wigner phase-space distribution function to study the density matrix, especially when we want to study the uncertainty products in detail [1,4].

For two coordinate variables, the Wigner function is defined as [4]

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \int \exp\left\{-2i(p_1 y_1 + p_2 y_2)\right\}$$

$$\times \psi^*(x_1 + y_1, x_2 + y_2)\psi(x_1 - y_1, x_2 - y_2)dy_1 dy_2. \qquad (4.1)$$

The Wigner function corresponding to the wave function of Eq.(3.2) is

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \exp\left\{-e^\eta(x_1\cos\frac{\alpha}{2} - x_2\sin\frac{\alpha}{2})^2 - e^{-\eta}(x_1\sin\frac{\alpha}{2} + x_2\cos\frac{\alpha}{2})^2\right.$$

$$\left. -e^{-\eta}(p_1\cos\frac{\alpha}{2} - p_2\sin\frac{\alpha}{2})^2 - e^\eta(p_1\sin\frac{\alpha}{2} + p_2\cos\frac{\alpha}{2})^2\right\}. \qquad (4.2)$$

If we do not make observations in the $x_2 p_2$ coordinates, the Wigner function becomes

$$W(x_1, p_1) = \int W(x_1, x_2; p_1, p_2)dx_2 dp_2. \qquad (4.3)$$

The evaluation of the integral leads to

$$W(x_1, x_2; p_1, p_2) = \left\{\frac{1}{\pi^2(1 + \sinh^2\eta\sin^2\alpha)}\right\}^{1/2}$$

$$\times \exp\left\{-\left(\frac{x_1^2}{\cosh\eta - \sin\eta\cos\alpha} + \frac{p_1^2}{\cosh\eta + \sin\eta\cos\alpha}\right)\right\}. \qquad (4.4)$$

This Wigner function gives an elliptic distribution in the phase space of x_1 and p_1. This distribution gives the uncertainty product of

$$(\Delta x)^2(\Delta p)^2 = \frac{1}{4}(1 + \sinh^2\eta\sin^2\alpha). \qquad (4.5)$$

This expression becomes $1/4$ if the oscillator system becomes uncoupled with $\alpha = 0$. Because x_1 is coupled with x_2, our ignorance about the x_2 coordinate, which in this case acts as Feynman's rest of the universe, increases the uncertainty in the x_1 world which, in Feynman's words, is the system in which we are interested.

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V. DENSITY MATRIX AND ENTROPY

Since the Wigner function is constructed from the density matrix, it is straightforward to show [4]

$$Tr[\rho(x_1, x_1)] = \int W(x_1, p_1)dx_1 dp_1, \tag{5.1}$$

and

$$Tr[\rho^2] = 2\pi \int W^2(x_1, p_1)dx_1 dp_1. \tag{5.2}$$

If we compute these integrals, $Tr(\rho) = 1$, as it should be for all pure and mixed states. On the other hand, $Tr(\rho^2)$ becomes

$$Tr(\rho^2) = 1/(1 + \sinh^2 \eta \sin^2 \alpha)^{1/2}, \tag{5.3}$$

which is in general less than one. This gives a measure of impurity and also the degree of the effect of our ignorance on the system in which we are interested.

Let us translate this into the language of the density matrix. If both x_1 and x_2 are measured, the density matrix is

$$\rho(x_1, x_2; x_1', x_2') = \psi(x_1, x_2)\psi^*(x_1', x_2'). \tag{5.4}$$

In terms of the expansion of the wave function given in Eq.(3.4),

$$\rho(x_1, x_2; x_1', x_2') = \sum_{n_1 n_2} \sum_{m_1 m_2} A_{m_1 m_2}(\alpha, \eta) A^*_{n_1 n_2}(\alpha, \eta)$$
$$\times \phi_{m_1}(x_1)\phi_{m_2}(x_2)\phi^*_{n_1}(x_1)\phi^*_{n_2}(x_2). \tag{5.5}$$

If both variables are measured, this is a pure-state density matrix.

On the other hand, if we do not make a measurement in the x_2 space, we have to construct the matrix $\rho(x_1, x_1')$ by taking the trace over the x_2 variable:

$$\rho(x_1, x_1') = \int \rho(x_1, x_2; x_1', x_2)dx_2. \tag{5.6}$$

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Then the density matrix $\rho(x_1, x_1')$ takes the form

$$\rho(x_1, x_1') = \sum_{m,n} C_{mn}(\alpha, \eta) \phi_m(x_1) \phi_n^*(x_1'), \tag{5.7}$$

with $C_{mn}(\alpha, \eta) = \sum_k A_{mk}(\alpha, \eta) A_{nk}^*(\alpha, \eta)$. The matrix $C_{mn}(\alpha, \eta)$ is also called the density matrix. The matrix C_{mn} is Hermitian and can therefore be diagonalized. If the diagonal elements are ρ_m, the entropy of the system is defined as [7,8]

$$S = -\sum_m \rho_m \ln(\rho_m). \tag{5.8}$$

The entropy is zero for a pure state, and increases as the system becomes impure. Like $Tr(\rho^2)$, this quantity measures the effect of our ignorance about the rest of the universe.

It is very important to realize that the above form of entropy can be defined irrespective of whether or not the system is in thermal equilibrium. As soon as we have the entropy, we are tempted to introduce the temperature. This is not always right. The temperature can only be introduced into the system in thermal equilibrium.

VI. PHYSICAL MODELS

There are many physical models based on coupled harmonic oscillators, such as the Lee model in quantum field theory [9], the Bogoliubov transformation in superconductivity [10], two-mode squeezed states of light [5,6,11], the covariant harmonic oscillator model for the parton picture [12], and models in molecular physics [13]. There are also models of current interest in which one of the variables is not observed, including thermo-field dynamics [14], two-mode squeezed states [15,16], the hadronic temperature [17], and the Barnet-Phoenix version of information theory [18]. They are indeed the examples of Feynman's rest of universe. In all of these cases, the mixing angle α is 90°, and the mathematics becomes much simpler. The Wigner function of Eq.(4.2) then becomes

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \exp\left\{-\frac{1}{2}\left[e^\eta(x_1 - x_2)^2 + e^{-\eta}(x_1 + x_2)^2\right.\right.$$
$$\left.\left. + e^{-\eta}(p_1 - p_2)^2 + e^\eta(p_1 + p_2)^2\right]\right\}. \tag{6.1}$$

This simple form of the Wigner function serves a starting point for many of the theoretical models including some mentioned above.

If the mixing angle α is 90°, the density matrix also takes a simple form. The wave function of Eq.(3.2) becomes

$$\psi_\eta(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{4}\left[e^\eta(x_1 - x_2)^2 + e^{-\eta}(x_1 + x_2)^2\right]\right\}. \tag{6.2}$$

As was discussed in the literature for several different purposes [4,20,21], this wave function can be expanded as

$$\psi_\eta(x_1, x_2) = \frac{1}{\cosh \eta} \sum_k (\tanh \eta)^k \phi_k(x_1)\phi_k(x_2). \tag{6.3}$$

From this wave function, we can construct the pure-state density matrix

$$\rho_\eta(x_1, x_2; x_1', x_2') = \psi_\eta(x_1, x_2)\psi_\eta(x_1', x_2'), \tag{6.4}$$

which satisfies the condition $\rho^2 = \rho$:

$$\rho_\eta(x_1, x_2; x_1', x_2') = \int \rho_\eta(x_1, x_2; x_1'', x_2'')\rho_\eta(x_1'', x_2''; x_1', x_2')dx_1''dx_2''. \tag{6.5}$$

If we are not able to make observations on the x_2, we should take the trace of the ρ matrix with respect to the x_2 variable. Then the resulting density matrix is

$$\rho_\eta(x_1, x_1') = \int \psi_\eta(x_1, x_2) \left\{\psi_\eta(x_1', x_2)\right\}^* dx_2. \tag{6.6}$$

If we complete the integration over the x_2 variable,

$$\rho_\eta(x_1, x_1') = \left(\frac{1}{\pi \cosh(2\eta)}\right)^{1/2} \exp\left\{-\frac{1}{4}[(x_1 + x_1')^2/\cosh(2\eta) + (x_1 - x_1')^2 \cosh 2\eta]\right\}. \tag{6.7}$$

The diagonal elements of the above density matrix is

$$\rho(x_1, x_1) = \left(\frac{1}{\pi \cosh 2\eta}\right)^{1/2} \exp\left(-x_1^2/\cosh 2\eta\right). \tag{6.8}$$

With this expression, we can confirm the property of the density matrix: $Tr(\rho) = 1$. As for the trace of ρ^2, we can perform the integration

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$$Tr\left(\rho^2\right) = \int \rho_\eta(x_1, x_1')\rho_\eta(x_1', x_1)dx_1'dx_1$$

$$= \left(\frac{1}{\cosh\eta}\right)^2, \tag{6.9}$$

which is less than one for nonzero values of η.

The density matrix can also be calculated from the expansion of the wave function given in Eq.(6.3). If we perform the integral of Eq.(6.6), the result is

$$\rho_\eta(x_1, x_1') = \left(\frac{1}{\cosh\eta}\right)^2 \sum_k (\tanh\eta)^{2k}\phi_k(x_1)\phi_k^*(x_1'), \tag{6.10}$$

which leads to $Tr(\rho) = 1$. It is also straight-forward to compute the integral for to $Tr(\rho^2)$. The calculation leads to

$$Tr\left(\rho^2\right) = (1/\cosh\eta)^4 \sum_k (\tanh\eta)^{4k}. \tag{6.11}$$

The sum of this series is $(1/\cosh\eta)^2$, which is the same as the result of Eq.(6.9).

This is of course due to the fact that we are not making measurement on the x_2 variable. The standard way to measure this ignorance is to calculate the entropy defined as [8]

$$S = -Tr\left(\rho\ln(\rho)\right). \tag{6.12}$$

If we use the density matrix given in Eq.(6.10), the entropy becomes

$$S = 2\left\{(\cosh\eta)^2 \ln(\cosh\eta) - (\sinh\eta)^2 \ln(\sinh\eta)\right\}. \tag{6.13}$$

This expression can be translated into a more familiar form if we use the notation

$$\tanh\eta = \exp\left(-\frac{\omega}{kT}\right). \tag{6.14}$$

The ratio ω/kT is a dimensionless variable. In terms of this variable, the entropy takes the form

$$S = \left(\frac{\omega}{kT}\right)\frac{1}{\exp(\omega/kT) - 1} - \ln\left[1 - \exp(-\omega/kT)\right]. \tag{6.15}$$

This is the entropy for a system of harmonic oscillators in thermal equilibrium. Thus, for this oscillator system, we can relate our ignorance to the temperature.

VII. CONCLUDING REMARKS

It is interesting to note that Feynman's rest of the universe appears as an increase in uncertainty and entropy in the system in which we are interested. In the case of coupled oscillators, the entropy allows us to introduce the variable which can be associated with the temperature. The density matrix is the pivotal instrument in evaluating the entropy. At the same time, the Wigner function is convenient for evaluating the uncertainty product. We can see clearly from the Wigner function how the ignorance or the increase in entropy increases the uncertainty in measurement.

The major strength of the coupled oscillator problem is that its classical mechanics is known to every physicist. Not too well known is the fact that this simple device has enough symmetries to serve as an analog computer for many of the current problems in physics. Indeed, this simple system can accommodate the symmetries contained in O(3,3) which is the group of Lorentz transformations applicable to three space-like and three time-like dimensions [2]. The group O(3,3) is has many interesting subgroups. Many, if not most, of the symmetry groups in physics are subgroups of this O(3,3) group.

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